A Sequence of Beurling Functions Related to the Natural Approximation B_n Defined by an Iterative Construction Generating Square-Free Numbers k_i and the Value of the Möbius Function $\mu(k_i)$

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Abstract. For a function $F_n(x) = \sum_{k=1}^n a_k\, \rho\left(\frac{\theta_k}{x}\right)$, where $\rho(x) \coloneqq x - [x]$, with $a_k \in \mathbb{C}$ and $0 < \theta_k \leqslant 1$ satisfying $\sum_{k=1}^n a_k\, \theta_k = 0$, is known to have

$$\left|\frac{1}{s}\left(1 - \zeta(s)\sum_{k=1}^{n} a_k\, \theta_k^s\right)\right| = \left|\int_0^1 (F_n(x)+1)\, x^{s-1}\, dx\right| \leqslant \|F_n(x)+1\| \, \left\|x^{s-1}\right\|;$$

where the first relation follows from straightforward computation and the second using Schwarz inequality in $L^2([0,1], dx)$. Therefore, if the first norm in the right hand side above would be arbitrarily small for a suitable choice of n, a_k's and θ_k's the function $\zeta(s)$ would have zeros for Re $s > 1/2$. This is the Beurling approach to Riemann Hypothesis. Several *approximating sequences* were proposed, between them

$$B_n(x) \coloneqq \sum_{k=1}^{n} \mu(k)\, \rho\left(\frac{1/k}{x}\right) \;-\; n\left(\sum_{k=1}^{n} \frac{\mu(k)}{k}\right) \rho\left(\frac{1/n}{x}\right).$$

In the present work we construct *iteratively* a sequence of numbers $\{k_n\}$ and approximating functions $\{\widetilde{B}_n\}$ converging *pointwise* to -1 in $[0,1]$. We prove results which seems to suggest the relation $\widetilde{B}_n = B_n$ and we conjecture that a sufficient condition for this is that the set $\{k_n\}_{n\in\mathbb{N}}$ be equal to the set of square-free numbers, $\mathbb{M} \coloneqq \{m \in \mathbb{N} : \mu(m) \neq 0\}$. Numerical evidence seems to support both conjectures.

Anyway, we think that these sequences are interesting by itself because our construction not only generates square-free (hence prime) numbers k_i, but also the value of the Möbius function $\mu(k_i)$. Our definition is independent of $\mathbb{M}$ and μ, with the k_i's arising as discontinuity points of the $\widetilde{B}_i$'s.

As for the case of B_n, we prove that sequence $\widetilde{B}_n$ is not convergent to -1 in $L^2([0,1], dx)$. Consequently, we focus our analysis not on L^2 norm at right hand side in first expression above but on the integral at the middle term. This procedure seems to be useful to elucidate the lack of L^2 convergence for *step* Beurling functions.

1 Introduction: Beurling Functions and Approximating Sequences

Denote as $[x]$ the integer part of x, i.e. the greatest integer less than or equal to x and define the fractional part function by $\rho(x) = x - [x]$. Given $n \in \mathbb{N}$ and two families of parameters $\{a_k\}_{k=1}^n \subset \mathbb{C}$ and $\{\theta_k\}_{k=1}^n \subset (0, 1]$, we define a *Beurling function* as a function $F = F_n$ (the sub-index n included in the notation for convenience) of the form

$$F_n(x) := \sum_{k=1}^{n} a_k \, \rho\left(\frac{\theta_k}{x}\right), \tag{1}$$

For a Beurling function F_n, an elementary computation shows

$$\int_0^1 (F_n(x) + 1)\, x^{s-1}\, dx = \frac{\sum_{k=1}^n a_k\, \theta_k}{s-1} + \frac{1}{s}\left(1 - \zeta(s) \sum_{k=1}^{n} a_k\, \theta_k^s\right); \quad \operatorname{Re} s > 0. \tag{2}$$

See, for instance, [1, p. 253] for a proof. It is useful, but not always necessary, assume that the parameters defining the function F_n satisfy the additional condition

$$\sum_{k=1}^{n} a_k\, \theta_k = 0. \tag{3}$$

In this case, first term at right hand side of (2) vanishes, simplifying the expression. Identity (2) is the starting point of a theorem by Beurling; see [1, p. 252] for a proof and further references. Here we just remark that an elementary computation, using Schwarz inequality for the integral at left hand side of (2), allows to show that a sufficient condition for Riemann Hypothesis (RH) is that $\|F(x) + 1\|$ be done arbitrarily small for a suitable choice of n, a_k's and θ_k's, where $\|.\|$ denotes the norm in $L^2([0, 1], dx)$. We will refer to this last condition as to the *Beurling criterion* (BC) for RH. It was proved in [3] that BC remains unchanged if we restrict the parameters θ_k to be reciprocal of natural numbers, i.e. $\theta_k = 1/b_k$, with $b_k \in \mathbb{N}$.

Several *approximating* functions (1) were proposed. From (2) and (3), we have that under BC the "partial sum"

$$\sum_{k=1}^{n} a_k\, \theta_k^s. \tag{4}$$

is an approximation to the inverse Riemann Zeta Function $1/\zeta(s)$, which is known to have an expression as a Dirichlet series

$$\frac{1}{\zeta(s)} = \sum_{k=1}^{n} \frac{\mu(k)}{k^s}, \tag{5}$$

convergent for $\operatorname{Re} s > 1$. Therefore, a (naive) first choice for an approximating function would be

$$S_n(x) := \sum_{k=1}^{n} \mu(k)\, \rho\left(\frac{1/k}{x}\right). \tag{6}$$

But this function does not matches the condition (3). We can handle this without subtlety, just taking out the difference, given by $g(n)$, where

$$g(t) := \sum_{\mathbb{N} \ni k \leqslant t} \frac{\mu(k)}{k}. \tag{7}$$

Therefore a second choice would be

$$B_n(x) := \sum_{k=1}^{n} \mu(k)\, \rho\left(\frac{1/k}{x}\right) - n\, g(n)\, \rho\left(\frac{1/n}{x}\right). \tag{8}$$

$$= \sum_{k=1}^{n-1} \mu(k)\, \rho\left(\frac{1/k}{x}\right) - n\, g(n-1)\, \rho\left(\frac{1/n}{x}\right). \tag{9}$$

There are also variants on the same theme, as

$$V_n(x) := \sum_{k=1}^{n} \mu(k)\, \rho\left(\frac{1/k}{x}\right) - g(n)\, \rho\left(\frac{1}{x}\right). \tag{10}$$

Sequences (6), (9) and (10) are known to be *not* convergent to -1 in $L^2([0,1], dx)$, as proved in [2].

2 From Integrals to Series

The integral in (2) can be expressed alternatively as a series, by a "change of variable" under suitable hypothesis. A Beurling function constant between the reciprocal of the natural numbers will be called a *step Beurling function*, i.e. such a function takes (non-necessarily different) constant values in each of the intervals $(\frac{1}{k+1}, \frac{1}{k}]$, for all $k \in \mathbb{N}$. Note that Beurling functions are left-continuous, because $\rho(x)$ is right-continuous.

Lemma 1. *Let F_n be a step Beurling function, such that $F_n(x) = -1$ if $x \in (\frac{1}{m}, 1]$, where $m \in \mathbb{N}$. (If $m = 1$ this is an empty condition, therefore in this case there is not additional condition at all). Define $f_n(k) := F_n(1/k)$, for $k \in \mathbb{N}$. Then,*

$$s \int_0^1 (F_n(x) + 1)\, x^{s-1}\, dx \;=\; \sum_{k=m}^{\infty} f_n(k) \left[\frac{1}{k^s} - \frac{1}{(k+1)^s} \right] \;+\; \frac{1}{m^s}. \tag{11}$$

Proof:

$$\int_0^1 (F_n(x) + 1)\, x^{s-1}\, dx \;=\; \int_0^{\frac{1}{m}} (F_n(x) + 1)\, x^{s-1}\, dx \;=\; \sum_{k=m}^{\infty} \int_{\frac{1}{k+1}}^{\frac{1}{k}} (F_n(x) + 1)\, x^{s-1}\, dx$$

$$= \sum_{k=m}^{\infty} (F_n(1/k) + 1) \int_{\frac{1}{k+1}}^{\frac{1}{k}} x^{s-1}\, dx \;=\; \sum_{k=m}^{\infty} (F_n(1/k) + 1)\, \frac{x^s}{s} \Big|_{\frac{1}{k+1}}^{\frac{1}{k}}$$

$$= \frac{1}{s} \sum_{k=m}^{\infty} (F_n(1/k) + 1) \left[\frac{1}{k^s} - \frac{1}{(k+1)^s} \right] \;=\; \frac{1}{s} \sum_{k=m}^{\infty} F_n(1/k) \left[\frac{1}{k^s} - \frac{1}{(k+1)^s} \right] \;+\; \frac{1}{s} \frac{1}{m^s}. \quad \blacksquare$$

Remarks. **(1)** Observe that a step Beurling function $F_n(x)$ is completely determined for $x \in [0, 1]$ by the arithmetic function $f_n(k)$, $k \in \mathbb{N}$. This arithmetic function can be extended, just by defining $f_n(x) := F_n(1/x)$, for $x \in \mathbb{R}$, becoming right-continuous.

(2) Just for reference, we define an *arithmetic Beurling function* as a right-continuous function f on $[1, +\infty)$ constant between the natural numbers, such that $f(1/x)$ is a Beurling function. Thus, there exists a correspondence between step and arithmetic Beurling functions and the integrals involving the former correspond to series involving the later as expressed in (11).

(3) We can think in relation (11) as a "change of variables" in the integral, turning the integration domain from $[0, 1]$ to $[1, +\infty)$. This can be visualized also like to put a zoom on the original integration domain, reflecting the interval $[0, 1]$ and then stretching it to fit on $[1, +\infty)$.

3 An Arithmetic Beurling Function Iteratively Defined

3.1 Beurling Binomials

One of the simplest arithmetic Beurling functions matching (3) are given by

$$\beta_{a,b}(x) := \rho\left(\frac{x}{a}\right) \; - \; \frac{b}{a}\,\rho\left(\frac{x}{b}\right), \tag{12}$$

when $a, b \in \mathbb{N}$. These "binomials" will be the basic blocks in our construction, thus we summarize some of its elementary properties in the following result.

Lemma 2. *Consider $a, b \in \mathbb{R}$, with $0 < a < b$. Then,*

- **a.** $\rho\left(\frac{x}{a}\right)$ *and* $\rho\left(\frac{x}{b}\right)$ *are right-continuous, and linearly independent functions.*
- **b.** $\beta_{a,b}(x) = 0$, *when* $0 \leqslant x < a$.
- **c.** *Let $k \in \mathbb{N}$ be such that $(k-1)a < b \leqslant ka$. Then,*

$$\beta_{a,b}(x) = \begin{cases} -j & \text{if } ja \leqslant x < (j+1)a, \text{ for } j = 1, \ldots, (k-2); \\ -(k-1) & \text{if } (k-1)a \leqslant x < b. \end{cases}$$

- **d.** *Assume $a, b \in \mathbb{N}$. Then, $\beta_{a,b}(x)$ is constant when $k \leqslant x < k+1$, for all $k \in \mathbb{N}$.*

3.2 The Aproximating Sequence $\widetilde{B}_i$

We will define a sequence of numbers $\{k_i\}$ and functions $\{b_i\}$ iteratively as follows. We start with the definition

$$\begin{aligned} k_1 &:= 1; \\ k_2 &:= 2; \\ b_2(x) &:= \rho\left(\frac{x}{k_1}\right) \; - \; \frac{k_2}{k_1}\,\rho\left(\frac{x}{k_2}\right). \end{aligned} \tag{13}$$

And for $i \geqslant 2$ define $k_{i+1} := k_i + j$, where j is the less integer such that $b_i(k_i + j) \neq b_i(k_i)$, and

$$b_{i+1}(x) := b_i(x) \; + \; (1 + b_i(k_i))\left[\rho\left(\frac{x}{k_i}\right) \; - \; \frac{k_{i+1}}{k_i}\,\rho\left(\frac{x}{k_{i+1}}\right)\right]. \tag{14}$$

Observe that each b_i is a linear combination of $\beta_{p,q}$. Other elementary properties are given in the next result, which is a direct consequence of Lemma 2.

Lemma 3. *For any $i \in \mathbb{N}$ we have*

- **a.** *b_i is an arithmetic Beurling function, i.e. a right-continuous function constant between the natural numbers.*
- **b.** $b_{i+1}(k_i) = -1$.
- **c.** *Assume $k_{i+1} \leqslant 2k_i$ for $i \geqslant 2$. Then, $b_i(x) = -1$ for all $x \in [1, k_i)$. In particular, the sequence $\{b_i\}_{i\in\mathbb{N}}$ converges pointwise to -1 in $[1, +\infty)$.*

Denote $\widetilde{B}_n(x) := b_n(1/x)$. As in the case of B_n we have the following result.

Lemma 4. *$\widetilde{B}_n$ do not converges to -1 in $L^2([0,1], dx)$.*

Proof: Denoting $e_k(x) = \rho\left(\frac{1}{kx}\right) - \frac{1}{k}\,\rho\left(\frac{1}{x}\right)$, we have $\beta_{p,q}(1/x) = e_p(x) - \frac{q}{p}\, e_q(x)$. Therefore, each $\widetilde{B}_n$ is a (finite) linear combination of e_k and the statement of the lemma follows from Proposition 4.7 in [2]. ∎

4 Relation Between $\widetilde{B}_n$ and B_n

The next result is relevant in order to establish a relation between the sequence $\{k_i\}$ and the square-free numbers and, on the other hand, between $\widetilde{B}_n$ and B_n.

Lemma 5. *The following conditions are equivalent*

- **a.** $\displaystyle\sum_{j=1}^{i} \frac{\mu(k_j)}{k_j} = \frac{1 + b_i(k_i)}{k_i}$, *for $i \geqslant 2$.*
- **b.** $\displaystyle b_i(x) = \sum_{j=1}^{i-1} \mu(k_j)\,\rho\left(\frac{x}{k_j}\right) - k_i \left(\sum_{j=1}^{i-1} \frac{\mu(k_j)}{k_j}\right) \rho\left(\frac{x}{k_i}\right)$, *for $i \geqslant 2$.*
- **c.** $\displaystyle\frac{\mu(k_i)}{k_i} = \frac{1 + b_i(k_i)}{k_i} - \frac{1 + b_{i-1}(k_{i-1})}{k_{i-1}}$, *for $i \geqslant 3$.*

Furthermore, if the condition $k_{i+1} < 2k_i$ is valid for $i \geqslant 2$, then all conditions above are also equivalent to the following ones

d. $\mu(k_{i+1}) = b_i(k_{i+1}) \;-\; b_i(k_i)$*, for* $i \geqslant 2$.

e. $\displaystyle\sum_{j=1}^{i} \mu(k_j) \left[\frac{k_i}{k_j}\right] = 1$*, for* $i \geqslant 1$.

Comparing (9) and expression in Lemma 5 (b) we can state the following conjecture.

Conjecture 1. $\widetilde{B}_n \stackrel{?}{=} B_n$*, for all* $n \in \mathbb{N}$.

Observe that $\widetilde{B}_n$ is not a subsequence of B_n, strictly speaking. Sequence $\{\widetilde{B}_i\}$ depends on the numbers $\mathbb{K} \coloneqq \{k_i\}_{i\in\mathbb{N}}$, the firsts of them are given by

$$\mathbb{K} = \{1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, \ldots\}.$$

These are all square-free numbers (incidentally, we prefer to denominate the numbers in $\mathbb{M} \coloneqq \{k \in \mathbb{N} \,:\, \mu(k) \neq 0\}$ as *Möbius numbers* rather than "square-free", because they are also cube-free, 4-th-power-free, etc.), and numerical evidence suggest $\mathbb{K} \subseteq \mathbb{M}$. Moreover, apparently none Möbius number is omitted, fact that seems to support the following conjecture.

Conjecture 2. $\mathbb{K} \stackrel{?}{=} \mathbb{M}$

If Conjecture 2 is true, then relation in Lemma 5 (e) is a well known result; see [4, p. 66]. It is also known that square-free numbers are distributed with density $6/\pi^2$; see [5, Thm. 333, p. 269]. We can estimate $k_{i+1} \approx k_i + \pi^2/6$, or $k_{i+1}/k_i \approx 1 + \pi^2/6k_i$ and this is less than 2 for $k_i > \pi^2/6 \approx 1.64$. Thus, condition $k_{i+1} < 2k_i$ seems to be reasonable also. This highly speculative argument seems to suggest that Conjecture 1 follows from 2.

5 Further Comments and Questions

Comparison of (11) and (2) (assuming (3)) gives

$$\sum_{k=m}^{\infty} F_n(1/k) \left[\frac{1}{k^s} - \frac{1}{(k+1)^s}\right] = \left(1 - \zeta(s) \sum_{k=1}^{n} a_k\, \theta_k^s\right) \;-\; \frac{1}{m^s}. \tag{15}$$

For the particular case of B_n, where $m = n$, this expression is given by

$$\sum_{k=n}^{\infty} B_n(1/k) \left[\frac{1}{k^s} - \frac{1}{(k+1)^s} \right] = (s-1)\zeta(s) \int_n^{\infty} \frac{g(t)}{t^s}\, dt \; - \; \frac{1}{n^s}. \tag{16}$$

In particular, $|g(t)| = \mathcal{O}(t^{-\frac{1}{2}})$ is a sufficient condition for RH. An old result by de la Vallée-Poussin states $|g(t)| = \mathcal{O}(\frac{1}{\ln t})$; see [6, p. 92].

If we apply Schwarz inequality in $l^2(\mathbb{N})$ to left hand side in (15) trying to get an analog of BC for step Beurling functions we have

$$\left| \left(1 - \zeta(s) \sum_{k=1}^{n} a_k\, \theta_k^s \right) \right| \leqslant \left| \frac{1}{m^s} \right| \; + \; \left(\sum_{k=m}^{\infty} |F_n(1/k)|^2 \right)^{\frac{1}{2}} \left(\sum_{k=m}^{\infty} \left| \frac{1}{k^s} - \frac{1}{(k+1)^s} \right|^2 \right)^{\frac{1}{2}}. \tag{17}$$

Now, $F_n(1/x)$ is a periodic function on the unbounded interval $[1, +\infty)$, thus for any N multiple of the period we have

$$\frac{a}{p}(N - m) \leqslant \sum_{k=m}^{N} |F_n(1/k)|^2 \leqslant \frac{a}{p} N, \tag{18}$$

where $p = p(n)$ is the period and $a = a(n) := \sum_{k=1}^{p} |F_n(1/k)|^2$. Therefore, the series for $\|F_n(1/k)\|_{l^2(\mathbb{N})}$ is divergent. This argument could explain the failure of L^2 convergence for general step Beurling functions and it would be possible to write down along these lines an alternative proof of Proposition 4.7 in [2].

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